decrease in pressure for high liquid-phase concentrations of propylene. The predictions of Zdonik (7) appear to be based upon an arbitrary extrapolation of the data of Reamer and Sage (4) rather than upon a thermodynamic analysis as used to establish Figure 1.


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## NOTATION

$f$ = fugacity, psia.
$P$ = total pressure, psia.
$R$ = gas constant
$T$ = absolute temperature, °R.
$\bar{v}$ = liquid-phase partial molar volume, cu.ft./lb.-mole
$v$ = molar liquid volume, cu.ft./lb.-mole
$x$ = liquid-phase mole fraction
$y$ = vapor-phase mole fraction
$\alpha$ = binary interaction constant (a function of temperature), lb.-mole/cu.ft.
$\Phi$ = volume fraction
$\phi$ = vapor-phase fugacity coefficient
$\gamma$ = liquid-phase activity coefficient

**Subscripts**

$i$ = component
$c$ = critical

**Superscripts**

$o$ = standard state
P0 = corrected to zero pressure

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# Comments on Diffusion in Membrane-Limited Blood Oxygenators


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Weissman (1) has recently considered oxygen transport to blood through a membrane in the limiting case of no hemodynamic boundary layer. As indicated there, this problem may be solved by use of a linear diffusion equation for the membrane plus a nonlinear diffusion equation for the blood caused by the oxygen capacity of the hemoglobin in the red blood cells. The purpose of this note is to show that the numerical solution by Weissman was unnecessarily complicated and that his results and conclusions may be obtained much more clearly and simply by proper writing of the describing equations and boundary conditions, which lead to an analytical solution.

The usual scheme of making separate balances on the membrane and on the blood with a matching condition at the blood-membrane interface will be followed. Thus, the mathematical problem consists of the steady state diffusion equation for the membrane in the transverse direction with a plug flow (no transverse gradients because no hemodynamic boundary layer resistance) balance for the blood. Such an approach will be followed here.

As much as possible, the original nomenclature of Weissman has been preserved. In addition, the relations between oxygen partial pressure and concentration in the membrane and in the plasma, as defined by Reneau et al. (2), will be used: $C^m{}_{O_2} = c_m P_{O_2}$, $C^p{}_{O_2} = c_1 P^p{}_{O_2}$.

The diffusion of oxygen through the membrane is described by

$$c_m D_{O_2,m} \frac{\partial^2 P_{O_2}}{\partial y^{*2}} = 0 \tag{1}$$

where $c_m$ is the oxygen solubility in the membrane, $P_{O_2}$ is the partial pressure in the membrane, and $y^* = y/t$ is the dimensionless distance measured from the membrane-blood interface to the membrane gas interface, $y = t$. The proper boundary conditions on this equation are

$$c_m P_{O_2}\big|_{y^*=1} = c_m P^\infty{}_{O_2} \tag{2}$$

and

$$c_m P_{O_2}\big|_{y^*=0} = \left[\frac{c_m}{c_1}\right] c_1 P^p{}_{O_2} \tag{3}$$

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where

$P^{\infty}{}_{O_2}$ = oxygen partial pressure in gas on outside of membrane

where $c_1$ is the oxygen solubility in the plasma and $P^{p}{}_{O_2}$ is the oxygen partial pressure in the plasma.

The plug flow balance for the blood along the axial length $z$ is

$$\left[ L \frac{L}{N} \overline{w} \right] \frac{d}{dz} |C^{T}{}_{O_2}| = L \left[ D_{O_2,m} \frac{\partial C^{m}{}_{O_2}}{\partial y} \bigg|_{y=0} \right] \tag{4}$$

where $C^{T}{}_{O_2}$ is the total oxygen concentration in the blood (plasma + hemoglobin) and $C^{m}{}_{O_2}$ is the oxygen concentration in the membrane. $C^{T}{}_{O_2}$ is usually known at the flow channel entrance and provides the necessary $z$ boundary condition. Also, $\overline{w}$ is the mean velocity and $L$ and $L/N$ are the cross-section dimensions.

Equation (4) can be written more conveniently in terms of partial pressure by the following substitution

$$C^{T}{}_{O_2} = C^{p}{}_{O_2} + C^{H}{}_{O_2}$$

where

$$C^{H}{}_{O_2} \equiv \frac{T_{O_2}}{100} g(P^{p}{}_{O_2})$$

$T_{O_2}$ is the maximum saturation concentration, and $g[P_{O_2}]$ is the function giving percent saturation.

Substituting these relations into Equation (4) and introducing Weismann's dimensionless variables and rearranging leads to

$$\frac{d}{dz^*} [P^{p}{}_{O_2} + \psi (P^{p}{}_{O_2})] = \frac{Nt}{L} \left[ \frac{c_m}{c_1} \right] \frac{\partial P_{O_2}}{\partial y^*} \bigg|_{y^*=0} \tag{5}$$

where

$$z^* = z\, D_{O_2,m}/t^2\, \overline{w}$$

and

$$\psi (P^{p}{}_{O_2}) \equiv \frac{T_{O_2}}{100\, c_1} [g(P^{p}{}_{O_2})]$$

Note that the "paradox" mentioned by Weismann (reference *1*, p. 629) that the oxygenation of the blood does not apparently depend on both $z^*$ and the parameter $(Nt/L)$ is a simple "a priori" observation of rewriting Equation (5) as

$$\frac{d}{d\left[\frac{Nt\, z^*}{L}\right]} [P^{p}{}_{O_2} + \psi (P^{p}{}_{O_2})] = \frac{c_m}{c_1} \frac{\partial P^{p}{}_{O_2}}{\partial y^*} \bigg|_{y^*=0}$$

which can always be done with no approximation.

Equations (1), (2), and (3) can be easily solved exactly to yield

$$P_{O_2} = P^{\infty}{}_{O_2} + [P^{p}{}_{O_2} - P^{\infty}{}_{O_2}] (1 - y^*) \tag{6}$$

Thus, $P_{O_2}$ $(y^*, z^*)$ is an implicit function of $z^*$ and exactly a linear function of $y^*$. This is the result of Weissman's Figure 4 (reference *1*, p. 629) which was obtained by him numerically and is also the same as his equation written as an "approximation."

The remainder of the solution can be found from Equation (5), using Equation (6) and the initial condition that

$$[P^{p}{}_{O_2} + \psi (P^{p}{}_{O_2})]_{z^*=0} \text{ is given:}$$

$$\frac{c_m}{c_1} \frac{Nt}{L} z^* = - \int_{(P^{p}{}_{O_2} - P^{\infty}{}_{O_2})_{z^*=0}}^{(P^{p}{}_{O_2} - P^{\infty}{}_{O_2})} \left[ 1 + \frac{d\psi(\lambda + P^{\infty}{}_{O_2})}{d\lambda} - \right] \frac{d\lambda}{\lambda} \tag{7}$$

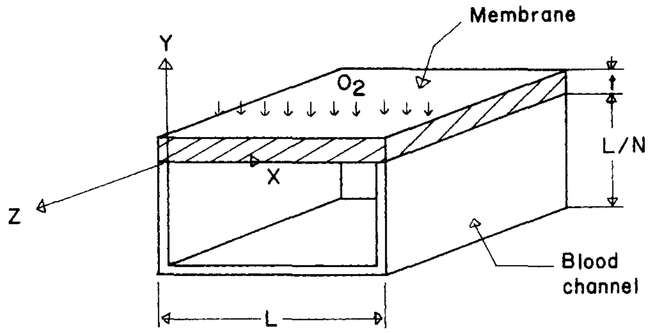


Fig. 1.

where $\lambda = [P^{p}{}_{O_2} - P^{\infty}{}_{O_2}]$

Thus, for a given $\psi(P_{O_2})$, Equation (7) gives the complete solution to the problem. In general Equation (7) would have to be evaluated numerically, but a quadrature is much simpler than numerical solution of partial differential equations.

For the special case of using an exponential function fit for $g(P^{p}{}_{O_2})$, and hence $\psi(P^{p}{}_{O_2})$, as used by Weissman and Mockros (*3*), an analytical solution in terms of tabulated exponential integrals can be found. For

$$\psi(P^{p}{}_{O_2}) = A_1 - A_1 e^{-\alpha_1 P^{p}{}_{O_2}} - A_2 e^{-\alpha_2 P^{p}{}_{O_2}} \tag{8}$$

$$= 63.7 \left( 100 - 99.9\, e^{-0.0368\, P^{p}{}_{O_2}} - 62.7\, e^{-0.032\, P^{p}{}_{O_2}} \right) \tag{8a}$$

the integral in Equation (7) becomes:

$$\frac{c_m}{c_1} \frac{Nt}{L} z^* = \ln \frac{(P^{\infty}_{O_2} - P^{p}_{O_2})_{z^*=0}}{(P^{\infty}_{O_2} - P^{p}_{O_2})}$$
$$+ \alpha_1 A_1 e^{-\alpha 1\, P^{\infty}_{O_2}} \{Ei\, [\alpha_1(P^{\infty}{}_{O_2} - P^{p}{}_{O_2})_0] - Ei\, [\alpha_1(P^{\infty}{}_{O_2} - P^{p}{}_{O_2})]\}$$
$$+ \alpha_2 A_2 e^{-\alpha 2\, P^{\infty}_{O_2}} \{Ei\, [\alpha_2(P^{\infty}{}_{O_2} - P^{p}{}_{O_2})_0] - Ei\, [\alpha_2(P^{\infty}{}_{O_2} - P^{p}{}_{O_2})]\} \tag{9}$$

where

$$Ei(x) \equiv \int_{-\infty}^{x} e^{\zeta} \frac{d\zeta}{\zeta}$$

and is tabulated (*4*).

Equation (9) can be (and has been) used to duplicate Weissman's Figure 3 (*1*) by solving for $P^{p}{}_{O_2}$, which can be converted to saturation increase from the initial condition and Equation (8).

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